

# Every étendue comes from a local equivalence relation

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## *Abstract*

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We first prove that, under suitable connectedness assumptions, the equivariant sheaves for a local equivalence relation on a space (or a locale) form an étendue topos. Our main result is that conversely, every étendue can be obtained in this way.

## Introduction

An étendue is a topos  $\mathcal{T}$  for which an object  $U \in \mathcal{T}$  exists such that  $U \rightarrow 1$  is epi and the slice topos  $\mathcal{T}/U$  is localic, that is,  $\mathcal{T}/U$  is equivalent to the category of sheaves on a locale. These étendue topoi were introduced by Grothendieck and Verdier [1, p. 478 ff.] in the context of foliations and local equivalence relations. It was suggested that for a suitable local equivalence relation  $r$  on a topological space, the category of  $r$ -invariant sheaves form an étendue topos. In this paper, we will consider the notion of a local equivalence relation  $r$  on a locale  $M$ . We will show that if  $r$  is locally simply connected (in an appropriate sense), then the category of  $r$ -invariant sheaves on  $M$  is a topos, and in fact an étendue. (We will also explain, in Example 2.3, how this result relates to a similar statement for local equivalence relations on topological spaces in [16].)

Our main result is that every étendue can be obtained this way. Indeed, in Theorem 7.1 we will show that for any étendue  $\mathcal{T}$ , there exists a local equivalence

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relation  $r$  on some locale  $M$  for which there is an equivalence of topoi

$$(*) \quad \text{Sh}(M, r) \approx \mathcal{T}.$$

Moreover, this local equivalence relation is locally simply connected in the sense referred to above. The construction of  $M$  and  $r$  is based on the observation that étendue topoi in some sense ‘classify’ local equivalence relations: every locally connected geometric morphism from a topos of sheaves on a locale  $M$  into an étendue  $\mathcal{T}$  gives rise to a canonical local equivalence relation on  $M$ . Furthermore, essential use is made of the construction from [6] of a localic cover with contractible fibres of any given topos.

For an étendue  $\mathcal{T}$  with enough points, there exists a *topological space* with a local equivalence relation  $r$  for which there is an equivalence of form  $(*)$ , but this space has to be obtained by a completely different construction, cf. [10, 11]; here the reader will also find a discussion of the relation between étendues, foliations, holonomy and monodromy.

Our main result is a presentation theorem for étendues; we wish to point out that this result bears no relation to the type of presentation considered in [9].

## 1. Equivalence relations on locales

Let  $X$  be a locale. By an equivalence relation on  $X$  we shall always mean a sublocale  $R \subseteq X \times X$  satisfying the usual conditions of reflectivity, symmetry, and transitivity, and in addition having the property that the two projection maps

$$d_0, d_1 : R \rightrightarrows X \tag{1.1}$$

are open maps. This implies that the coequalizer  $\pi_R : X \rightarrow X/R$  of  $d_0$  and  $d_1$  is also an open map [14]. Note that, unlike the case of topological spaces,  $R$  need not coincide with the kernel pair of  $\pi_R$ , cf. [8]. Since  $R$  is reflexive and transitive, there is a (truncated) simplicial complex of locales

$$R \times_X R \rightrightarrows R \rightrightarrows X. \tag{1.2}$$

By applying the functor  $\text{sh}(-)$ , we obtain a similar diagram of topoi and geometric morphisms. We write  $\text{sh}(X; R)$  for the associated descent topos. So the objects of  $\text{sh}(X; R)$  are sheaves  $E$  on  $X$  equipped with ‘descent data’  $\theta_E : d_0^* E \rightarrow d_1^* E$  satisfying a unit and cocycle condition. (This construction will be discussed in greater generality in Section 3.) Equivalently,  $\theta$  can be given in the form of an action by  $R$  on  $E$ , or of a transport on  $E$  along  $R$ , i.e. a map  $R \times_X E \rightarrow E$  satisfying usual associativity and unit laws. The localic reflection of this topos  $\text{sh}(X; R)$  is (the topos of sheaves on) the quotient locale  $X/R$ .

In the context of topological spaces, it is well known that  $\text{sh}(X; R)$  coincides with  $\text{sh}(X/R)$  in case the map  $d_0 : R \rightarrow X$  has ‘enough local sections’ (cf. [1, p.

480]). In fact, it is enough to require  $d_0$  to be an open map; the argument also works for locales:

**Proposition 1.1.** *For any equivalence relation  $R$  on a locale  $X$  (with  $d_0$  and  $d_1$  open), the map  $\text{sh}(X; R) \rightarrow \text{sh}(X/R)$  is an equivalence of topoi.*

**Proof.** It is enough to show that the topos  $\text{sh}(X; R)$  is generated by subobjects of the terminal object 1. Such subobjects are  $R$ -saturated open sublocales of  $X$ , i.e. sublocales of the form  $d_1 d_0^{-1}(U) \subseteq X$ , where  $U \subseteq X$  is any open sublocale. Such  $R$ -saturated sublocales carry a unique action by  $R$ , hence are objects of  $\text{sh}(X; R)$ . Now consider an arbitrary object  $E$  of  $\text{sh}(X; R)$ , given as an étale map  $p : E \rightarrow X$  with an action  $\theta : R \times_X E \rightarrow E$ . Let  $s : W \rightarrow E$  be any section of  $E \rightarrow X$  over an open  $W \subseteq X$ . We wish to show that  $W$  is covered by opens  $W_i \subseteq W$  with the property that  $s|_{W_i} : W_i \rightarrow E$  can be extended to an  $R$ -equivariant section

$$\tilde{s}_i : d_1 d_0^{-1}(W_i) \rightarrow E. \quad (1.3)$$

Thus each such section  $\tilde{s}_i$  is a map in the topos  $\text{sh}(X; R)$  from a subobject of 1 into  $E$ . All these sections  $\tilde{s}_i$ , for all possible sections  $s : W \rightarrow E$ , cover  $E$  since  $p : E \rightarrow X$  is a local homeomorphism. So this indeed shows that  $\text{sh}(X; R)$  is generated by subobjects of 1.

To construct these local extensions  $\tilde{s}_i$  from the given section  $s : W \rightarrow E$ , consider first the pullback  $d_1^*(E|_W) = E|_W \times_W R$  of  $E|_W \rightarrow W$  along  $d_1 : R|_W = R \cap (W \times W) \rightarrow W$ , as in

$$\begin{array}{ccc} d_1^*(E|_W) & \xrightarrow{\quad} & E|_W \\ \rho' \downarrow & & \downarrow p \\ R|_W = R \cap (W \times W) & \xrightarrow{d_1} & W \end{array} \quad (1.4)$$

The map  $p'$  in this diagram has two sections induced by  $s : W \rightarrow E$ , namely

$$s_1 = \theta(\text{id}, s d_0) : R|_W \rightarrow R \times_W E \rightarrow E, \quad s_2 = s d_1.$$

(In point-set notion,  $d_1^*(E|_W) = \{(x, y, e) \mid (x, y) \in R, x, y \in W, e \in p^{-1}(y)\}$ , and  $s_1(x, y) = \theta((x, y), s(x))$ ,  $s_2(x, y) = s(y)$ .) These two sections agree on the diagonal  $\Delta : W \rightarrow R|_W$ . Since  $p'$  is étale, it follows that they must agree on a neighbourhood  $N$  of the diagonal. We may assume that this neighbourhood is of the form

$$N = \bigcup_i R|_{W_i} = \bigcup_i R \cap (W_i \times W_i),$$

for some open cover  $W = \bigcup_i W_i$ . By definition of  $s_1$  and  $s_2$ , this means that each restriction  $s|_{W_i} : W_i \rightarrow E$  is  $(R|_{W_i})$ -equivariant. It follows that  $s|_{W_i}$  can be extended to the  $R$ -saturation  $d_1 d_0^{-1}(W_i)$  of  $W_i$ . Indeed, let  $P$  be the kernel pair of

$d_1 : d_0^{-1}(W_i) \rightarrow d_1 d_0^{-1}(W_i)$ , as in the diagram

$$P \xrightarrow[\pi_1]{\pi_0} d_0^{-1}(W_i) \rightarrow d_1 d_0^{-1}(W_i). \quad (1.5)$$

Since  $d_1 : R \rightarrow X$  is an open surjection by assumption, this diagram is a coequalizer, cf. [7]; furthermore, since  $s|_{W_i}$  is  $(R|_{W_i})$ -equivariant, the cocycle condition for the action  $\theta$  by  $R$  on  $E$  implies that the map  $s_1 : d_0^{-1}(W_i) \rightarrow E$ , given in point-set terms by  $s_1(x, y) = \theta((x, y), s(x))$ , satisfies the identity  $s_1 \pi_1 = s_1 \pi_2$ . Thus  $s_1$  factors through the coequalizer (1.5), to give the desired section  $\tilde{s}_i : d_1 d_0^{-1}(W_i) \rightarrow E$ .  $\square$

An equivalence relation  $R$  on a locale  $X$ , as above, is said to be *connected* (respectively *locally connected*) if  $d_0$  and  $d_1$  are connected (respectively locally connected) maps of locales, i.e. if the corresponding geometric morphisms  $d_0, d_1 : \text{sh}(R) \rightarrow \text{sh}(X)$  are connected, respectively locally connected.

**Proposition 1.2.** *If  $d_0, d_1$  are connected (respectively locally connected) maps, the quotient map  $X \rightarrow X/R$  is a connected (respectively locally connected) map.*

**Proof.** For the locally connected case, if  $d_0, d_1 : R \rightrightarrows X$  are locally connected, then by [15] so is the geometric morphism  $\text{sh}(X) \rightarrow \text{sh}(X; R)$ , and hence by Proposition 1.1,  $X \rightarrow X/R$  is a locally connected map. For the connected case, assume that  $d_0, d_1 : R \rightrightarrows X$  are connected. Again by Proposition 1.1, it suffices to see that  $\text{sh}(X) \rightarrow \text{sh}(X; R)$  is a connected geometric morphism. Consider two  $R$ -equivariant sheaves  $(E, \theta)$  and  $(F, \mu)$ , and a map  $\phi : E \rightarrow F$  in  $\text{sh}(X)$ . We must prove that  $\phi$  is  $R$ -equivariant, i.e. a map in  $\text{sh}(X; R)$ . Consider the two maps  $\alpha, \beta : d_0^* E \rightarrow d_0^* F$  described, in point-set notation, for  $(x, y) \in R$  and  $e \in E_x$ , by

$$\begin{aligned} \alpha((x, y), e) &= ((x, y), \mu((y, x), \phi(\theta((x, y), e)))) , \\ \beta((x, y), e) &= ((x, y), \phi(e)) . \end{aligned}$$

Thus  $\beta = d_0^*(\phi)$ , and since  $d_0^*$  is full and faithful,  $\alpha = d_0^*(\alpha')$  for a unique map  $\alpha' : E \rightarrow F$  in  $\text{sh}(X)$ . By the unit-condition for the actions  $\theta$  and  $\mu$ , we have  $\Delta^*(\alpha) = \Delta^*(\beta)$ , where  $\Delta : X \rightarrow R$  is the diagonal. Hence  $\alpha' = \Delta^* d_0^* \alpha' = \Delta^* \alpha = \Delta^* \beta = \phi$ , and thus, applying  $d_0^*$ ,  $\alpha = \beta$ . This identity expresses that  $\phi$  is an  $R$ -equivariant map. This proves the proposition.  $\square$

As a consequence, we obtain the following:

**Proposition 1.3.** *Let  $R$  be a connected equivalence relation on a locale  $X$ . Then for any sheaf  $E$  on  $X$ , there is at most one action by  $R$  on  $E$ .*

(If there is such an action, we call  $E$  an  *$R$ -invariant sheaf*.)

**Proof.** Let  $\theta$  and  $\theta'$  be two  $R$ -actions on  $E$ . Since the forgetful functor  $\text{sh}(X; R) \rightarrow \text{sh}(X)$  is full and faithful, the identity map on  $E$  in  $\text{sh}(X)$  must also be an  $R$ -equivariant map  $(E, \theta) \rightarrow (E, \theta')$ , thus  $\theta = \theta'$ .  $\square$

## 2. Local equivalence relations and sheaves

For a locale  $M$ , consider for each open  $U \subseteq M$  the set  $E_M(U)$  of equivalence relations  $R$  on  $U$ , as defined in Section 1. For open sublocales  $V \subseteq U \subseteq M$  there is an evident restriction map  $E_M(U) \rightarrow E_M(V)$ , making  $E_M$  into a presheaf on  $M$ . By definition [1, p. 485], a local equivalence relation on  $M$  is a global section of the associated sheaf  $\tilde{E}_M$ . An equivalence relation  $R$  on any locale  $U$  gives rise to a local equivalence relation  $L(R)$  on  $U$ . Let  $r$  be a local equivalence relation on  $M$ . An equivalence relation  $R$  on an open  $U \subseteq M$  will be called a *chart* for  $r$  if  $L(R)$  agrees with the restriction of  $r$  to  $U$ ; if  $V \subseteq U$ , then  $(V, R|_V)$  is also a chart for  $r$ ; we call it a *subchart* of  $(U, R)$ . An *atlas* for  $r$  is a family  $\{(U_i, R_i)\}$  of charts for  $r$  such that the  $U_i$ 's cover  $M$ . A family  $\{(U_i, R_i)\}$  will be an atlas for some local equivalence relation iff for any two indices  $i$  and  $j$ ,  $U_i \cap U_j$  is covered by open  $W$  such that  $R_i|_W = R_j|_W$ . An atlas is a *refinement* of another if each chart of the former is a subchart of some chart of the latter.

By our conventions in Section 1, it follows that any local equivalence relation  $r$  has an atlas consisting of charts  $(U, R)$  for which  $R \rightrightarrows U$  are open maps. Furthermore,  $r$  is said to be *locally connected* if any atlas for  $r$  can be refined by an atlas consisting of connected and locally connected charts, i.e. charts  $(U, R)$  for which  $R \rightrightarrows U$  are connected and locally connected maps. Such an atlas will be called a *connected atlas* for  $r$ .

Following [1], we now define, for a local equivalence relation  $r$  on a locale  $M$  and a sheaf  $F$  on  $M$ , the notion of an  $r$ -transport on  $F$ . Consider for an open  $U \subseteq M$  the set  $T_F(U)$  of pairs  $(R, \theta)$ , where  $R$  is an equivalence relation on  $U$  and  $\theta : R \times_U (F|_U) \rightarrow (F|_U)$  is an action by  $R$  on  $F|_U$  (as in Section 1). With the obvious restriction maps  $T_F(U) \rightarrow T_F(V)$  for opens  $V \subseteq U \subseteq M$ , this gives a presheaf  $T_F$  on  $M$ , with a projection map  $\pi : T_F \rightarrow E_M$ . Passing to the associated sheaves, we obtain a map  $\tilde{\pi} : \tilde{T}_F \rightarrow \tilde{E}_M$ . An  $r$ -transport on the sheaf  $F$  is by definition a global section  $t$  of  $\tilde{T}_F$  such that  $\tilde{\pi}(t) = r$ . A sheaf equipped with an  $r$ -transport is called an  *$r$ -invariant sheaf*, or an  *$r$ -sheaf*. Such an  $r$ -transport is thus given by an open cover  $\bigcup U_i = M$ , equivalence relations  $R_i$  on  $U_i$ , and actions  $\theta_i$  of  $R_i$  on  $F|_{U_i}$ , all locally compatible on intersections  $U_i \cap U_j$ . As before, we call  $\mathcal{U} = \{(U_i, R_i, \theta_i)\}$  an *atlas* for  $t$ , and each of its members a *chart* for  $t$ .

Any atlas or chart for  $t$  has an evident underlying atlas or chart for  $r$ . We note that if  $\bar{\mathcal{U}}$  is an atlas for  $t$  with underlying atlas  $\mathcal{U}$  for  $r$ , and  $\mathcal{V}$  is another atlas for  $r$  which refines  $\mathcal{U}$ , then  $\bar{\mathcal{U}}$  can be refined by an atlas  $\tilde{\mathcal{V}}$  for  $t$  which has the given  $\mathcal{V}$  as underlying atlas for  $r$ . It follows that for two sheaves with  $r$ -transport  $(F, t)$  and  $(F', t')$ , there exists atlases for  $t$  and  $t'$  with identical underlying atlas for  $r$ . It

also follows that if  $r$  is locally connected, any atlas for  $t$  can be refined by an atlas whose underlying atlas for  $r$  is connected.

In this paper, we shall only consider local equivalence relations  $r$  which are locally connected. For such an  $r$ , it follows readily from Proposition 1.2 that for a given sheaf  $F$  on  $M$ , there is at most one  $r$ -transport  $t$  on  $F$ . Thus (the existence of) an  $r$ -transport on  $F$  is a property, rather than an additional structure. For a locally connected  $r$ , we therefore define the category  $\mathrm{sh}(M, r)$  to be the full subcategory of  $\mathrm{sh}(M)$  consisting of sheaves on  $M$  which admit an  $r$ -transport (necessarily unique).

**Remark 2.1.** The property of being an  $r$ -sheaf on  $X$  is a local property. More explicitly, if  $q : Y \rightarrow X$  is an étale map (a local homeomorphism), then any local equivalence relation  $r$  on  $X$  induces, in an evident way, a local equivalence relation on  $Y$ , which we denote  $q^{\#}r$ ; if  $r$  is locally connected, then so is  $q^{\#}r$ ; and conversely, provided  $q$  is surjective. In this case, it is clear that if  $E \in \mathrm{sh}(X)$ , then  $E$  is an  $r$ -sheaf iff  $q^*E$  is an  $q^{\#}r$ -sheaf.

**Remark 2.2.** More generally, for an arbitrary local equivalence relation  $r$  on a locale  $M$  and two sheaves with  $r$ -transport  $(F, t)$  and  $(F', t')$ , there is a straightforward definition of transport-preserving map  $F \rightarrow F'$ , so that one obtains a category  $\mathrm{sh}(M, r)$ . Using the remarks in Section 1, one can easily show that in case  $r$  is locally connected, any sheaf map  $F \rightarrow F'$  is transport-preserving, so that for such  $r$ , the fact that the forgetful functor  $\mathrm{sh}(M, r) \rightarrow \mathrm{sh}(M)$  is full and faithful is a result, rather than a definition.

**Example 2.3.** For any locale  $M$ , there is a ‘maximal’ local equivalence relation  $r_{\max}$  on  $M$ , given by the single chart  $(U, R)$ , where  $U = M$  and  $R = M \times M$ . If  $M$  is a locally connected locale, then  $r_{\max}$  is also locally connected. The category  $\mathrm{sh}(M, r_{\max})$  is exactly the category of locally constant sheaves on  $M$ . This category is not in general a Grothendieck topos. For example [3, p. 314] when  $M$  is the Hawaiian earring,  $\mathrm{sh}(M, r_{\max})$  is not closed under infinite sums; on the other hand, if  $\mathrm{sh}(M, r_{\max})$  is a Grothendieck topos, it must have infinite sums, and these sums must be preserved by the forgetful functor, cf. loc. cit., Theorem 6; cf. also [10]. The fact that  $\mathrm{sh}(M, r_{\max})$  is not a Grothendieck topos disproves Theorem 4.14 in [16].

**Example 2.4.** Let  $r$  be a (locally connected) local equivalence relation on a locale  $M$ . For any locale  $T$ , there is a sheaf  $T^{(r)}$  on  $M$  of germs of  $r$ -invariant maps  $M \rightarrow T$ . A typical section of  $T^{(r)}$  over an open  $U \subseteq M$  is a map  $s : U \rightarrow T$  which has the property that  $U$  is covered by  $r$ -charts  $(U_i, R_i)$  such that each restriction  $s|_{U_i} : U_i \rightarrow T$  factors through the quotient map  $U_i \rightarrow R_i/R_i$ . This sheaf  $T^{(r)}$  has  $r$ -transport, hence is an object of  $\mathrm{sh}(M, r)$ . When  $T$  is the Sierpinski space,  $T^{(r)}$  is a subobject classifier for  $\mathrm{sh}(M, r)$ , and  $\mathrm{sh}(M, r)$  is an elementary topos. This is discussed more fully in [10].

### 3. Simplicial topoi and descent

Recall that a simplicial topos is a simplicial object  $\mathcal{E}_\bullet$  in the category of (Grothendieck) topoi, except that the simplicial identities are required to hold only up to coherent isomorphisms. Thus a simplicial topos consists of a sequence of topoi  $\mathcal{E}_n$  ( $n \geq 0$ ), and for each nondecreasing function  $\alpha : [n] \rightarrow [m]$  (where  $[n] = \{0, 1, \dots, n\}$ ) a geometric morphism

$$\mathcal{E}(\alpha) : \mathcal{E}_m \rightarrow \mathcal{E}_n ;$$

furthermore, for each such  $\alpha : [n] \rightarrow [m]$  and  $\beta : [m] \rightarrow [k]$ , there is given an isomorphism  $\theta_{\alpha, \beta} : \mathcal{E}(\alpha) \circ \mathcal{E}(\beta) \rightarrow \mathcal{E}(\beta\alpha)$ , and these  $\theta$ 's are required to satisfy suitable coherence conditions. (Thus, a simplicial topos is a homomorphism of bicategories from the category  $\Delta^{\text{op}}$  into the bicategory of Grothendieck topoi.)

We adopt the standard notation from simplicial sets; for example, we write  $d_j : \mathcal{E}_n \rightarrow \mathcal{E}_{n-1}$  for  $\mathcal{E}(\partial_j)$ , where  $\partial_j : [n-1] \rightarrow [n]$  is the strictly increasing function which omits  $j$  (for  $0 \leq j \leq n$ ).

For each simplicial topos  $\mathcal{E}_\bullet$  one can construct a universal augmentation  $\mathcal{D}(\mathcal{E}_\bullet)$ , as in

$$\mathcal{E}_2 \rightrightarrows \mathcal{E}_1 \rightrightarrows \mathcal{E}_0 \rightarrow \mathcal{D}(\mathcal{E}_\bullet) . \quad (3.1)$$

The category  $\mathcal{D}(\mathcal{E}_\bullet)$  can be explicitly described in various equivalent ways; e.g. as the category of *descent objects*: thus an object of  $\mathcal{D}(\mathcal{E}_\bullet)$  is a pair  $(E, \mu)$  where  $E$  is an object of  $\mathcal{E}_0$  and  $\mu : d_0^* E \rightarrow d_1^* E$  is an isomorphism satisfying the appropriate unit and cocycle conditions (cf. [15, section 3]); the arrows in  $\mathcal{D}(\mathcal{E}_\bullet)$  between two such objects  $(E, \mu)$  and  $(E', \mu')$  are arrows  $E \rightarrow E'$  in  $\mathcal{E}_0$  which are compatible with the 'descent data'  $\mu$  and  $\mu'$ . It follows from the general existence theorem for colimits of Grothendieck topoi ([15, Section 2] and [12]) that  $\mathcal{D}(\mathcal{E}_\bullet)$  is a Grothendieck topos, and is the colimit of the diagram  $\mathcal{E}_\bullet$ . The augmentation geometric morphism  $a : \mathcal{E}_0 \rightarrow \mathcal{D}(\mathcal{E}_\bullet)$  has as its inverse image the forgetful functor  $a^* : \mathcal{D}(\mathcal{E}_\bullet) \rightarrow \mathcal{E}_0$ , so that  $a^*(E, \mu) = E$ .

The following is part of [15, Theorem 3.6]:

**Lemma 3.1.** *For a simplicial topos  $\mathcal{E}_\bullet$ , if all the face maps  $d_j : \mathcal{E}_n \rightarrow \mathcal{E}_{n-1}$  are open (respectively locally connected, or atomic), then so is the augmentation  $a : \mathcal{E}_0 \rightarrow \mathcal{D}(\mathcal{E}_\bullet)$ .  $\square$*

In particular, if  $X_\bullet$  is a simplicial locale, we obtain a simplicial topos  $\text{sh}(X_\bullet)$  by constructing the topos of sheaves  $\text{sh}(X_n)$  on each locale  $X_n$ , and hence a descent topos  $\mathcal{D}(\text{sh}(X_\bullet))$ , and Lemma 3.1 gives the following:

**Lemma 3.2.** *For a simplicial locale  $X_\bullet$  in which all the face maps  $d_j : X_n \rightarrow X_{n-1}$*

are étale, the augmentation  $\mathrm{sh}(X_0) \rightarrow \mathcal{D}(\mathrm{sh}(X_*))$  is an atomic geometric morphism, and  $\mathcal{D}(\mathrm{sh}(X_*))$  is an étendue.

**Proof.** Since the  $d_j$  are étale, the induced geometric morphisms  $d_j : \mathrm{sh}(X_n) \rightarrow \mathrm{sh}(X_{n-1})$  are atomic. By Lemma 3.1, the augmentation  $\mathrm{sh}(X_0) \rightarrow \mathcal{D}(\mathrm{sh}(X_*))$ , which is evidently surjective, must also be atomic. Since this augmentation is also clearly a localic geometric morphism, it must be a slice, and thus  $\mathcal{D}(\mathrm{sh}(X_*))$  is an étendue.  $\square$

A map of simplicial topoi  $f : \mathcal{F} \rightarrow \mathcal{E}$  is given by geometric morphisms  $f_n : \mathcal{F}_n \rightarrow \mathcal{E}_n$  for each  $n \geq 0$ , together with, for each  $\alpha : [n] \rightarrow [m]$ , an isomorphism

$$f_\alpha : f_n \circ \mathcal{F}(\alpha) \rightarrow \mathcal{E}(\alpha) \circ f_m,$$

and these isomorphisms are required to be compatible with the isomorphisms  $\theta_{\alpha,\beta}$  for  $\mathcal{E}$  and  $\mathcal{F}$ . Such a map  $f : \mathcal{F} \rightarrow \mathcal{E}$  induces a geometric morphism  $\mathcal{D}(f) : \mathcal{D}(\mathcal{F}_*) \rightarrow \mathcal{D}(\mathcal{E}_*)$  between descent topoi, which is compatible with the augmentations in the sense that the square

$$\begin{array}{ccc} \mathcal{F}_0 & \xrightarrow{f_0} & \mathcal{E}_0 \\ a_{\mathcal{F}} \downarrow & & \downarrow a_{\mathcal{E}} \\ \mathcal{D}(\mathcal{F}_*) & \xrightarrow{\mathcal{D}(f)} & \mathcal{D}(\mathcal{E}_*) \end{array} \quad (3.2)$$

commutes up to canonical isomorphism. Later we will use the following lemma concerning connected geometric morphisms (these are morphisms whose inverse image functor is full and faithful).

**Lemma 3.3.** *Let  $f : \mathcal{F} \rightarrow \mathcal{E}$  be a map of simplicial topoi. If  $f_0$  is connected and  $f_1$  is surjective, then the induced geometric morphism  $\mathcal{D}(f) : \mathcal{D}(\mathcal{F}_*) \rightarrow \mathcal{D}(\mathcal{E}_*)$  is again connected.*

**Proof.** Consider two objects  $(E, \mu)$  and  $(E', \mu')$  in  $\mathcal{D}(\mathcal{E}_*)$ . We wish to show that arrows  $(E, \mu) \rightarrow (E', \mu')$  in  $\mathcal{D}(\mathcal{E}_*)$  correspond bijectively to arrows  $\mathcal{D}(f)^*(E, \mu) \rightarrow \mathcal{D}(f)^*(E', \mu')$  in  $\mathcal{D}(\mathcal{F}_*)$ . Since  $f_0^*$  is full and faithful by assumption, it evidently suffices to show that for an arrow  $\alpha : E \rightarrow E'$  in  $\mathcal{E}_0$ ,  $\alpha$  is compatible with descent data  $\mu : d_0^* E \rightarrow d_1^* E$  and  $\mu' : d_0^* E' \rightarrow d_1^* E'$  (in  $\mathcal{E}_1$ ) iff  $f_0^*(\alpha)$  is compatible with the induced descent data (in  $\mathcal{F}_1$ )

$$f_1^*(\mu) : d_0^* f_0^*(E) \cong f_1^* d_0^*(E) \rightarrow f_1^* d_1^*(E) \cong d_1^* f_0^*(E)$$

on  $f_0^*(E)$  and (similarly)  $f_1^*(\mu')$  on  $f_0^*(E')$ . But this readily follows by the assumption that  $f_1^* : \mathcal{E}_1 \rightarrow \mathcal{F}_1$  is a faithful functor.  $\square$



Recall that by the existence theorem for colimits [12, 15], already used in the construction of descent topoi  $\mathcal{D}(\mathcal{E})$ , the pushout topos  $\mathcal{B} \cup_{\mathcal{A}} \mathcal{C}$  of any two geometric morphisms  $f : \mathcal{A} \rightarrow \mathcal{B}$  and  $g : \mathcal{A} \rightarrow \mathcal{C}$  between Grothendieck topoi exists,

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{g} & \mathcal{C} \\ f \downarrow & & \downarrow v \\ \mathcal{B} & \xrightarrow{u} & \mathcal{B} \cup_{\mathcal{A}} \mathcal{C} \end{array} \quad (3.3)$$

and can be constructed simply as follows: the objects of  $\mathcal{B} \cup_{\mathcal{A}} \mathcal{C}$  are triples  $(B, C, v)$ , where  $B$  is an object of the topos  $\mathcal{B}$  and  $C$  one of  $\mathcal{C}$ , while  $v : f^*(B) \rightarrow g^*(C)$  is an isomorphism in the topos  $\mathcal{A}$ . An arrow  $(B, C, v) \rightarrow (B', C', v')$  in the pushout topos  $\mathcal{B} \cup_{\mathcal{A}} \mathcal{C}$  is given by a pair of arrows  $\beta : B \rightarrow B'$  in  $\mathcal{B}$  and  $\gamma : C \rightarrow C'$  in  $\mathcal{C}$  such that  $v' \circ f^*(\beta) = g^*(\gamma) \circ v$  in  $\mathcal{A}$ . In the square (3.3), the inverse images  $u^*$  and  $v^*$  of the indicated geometric morphisms are the evident forgetful functors.

One can easily verify that for a pushout square,  $u^*$  is full and faithful whenever  $g^*$  is; in other words, we have the following:

**Lemma 3.4.** *The pushout of a connected geometric morphism along any other geometric morphism is again connected ('connectedness is preserved under co-base-change').*  $\square$

Slightly more involved is the following lemma:

**Lemma 3.5.** *Let  $f : \mathcal{F} \rightarrow \mathcal{E}$  be a map of simplicial topoi, with induced geometric morphism  $\mathcal{D}(f) : \mathcal{D}(\mathcal{F}) \rightarrow \mathcal{D}(\mathcal{E})$ . If  $f_1 : \mathcal{F}_1 \rightarrow \mathcal{E}_1$  is connected and  $f_2 : \mathcal{F}_2 \rightarrow \mathcal{E}_2$  is surjective, then the square (3.2) is a pushout.*

**Proof.** Let us write  $\mathcal{P}$  for the pushout topos. Then the objects of  $\mathcal{P}$  are of the form

$$(F, \mu, E, v), \quad (3.4)$$

where  $F$  is an object of  $\mathcal{F}_0$  with descent data  $\mu : d_0^* F \rightarrow d_1^* F$ , while  $E$  is an object of  $\mathcal{E}_0$  and  $v : F \rightarrow f_0^* E$  is an isomorphism. This gives an arrow  $f_1^* d_0^* E \rightarrow f_1^* d_1^* E$  in  $\mathcal{F}_1$ : the broken arrow in the following diagram

$$\begin{array}{ccc} f_1^* d_0^* E \cong d_0^* f_0^* E \cong d_0^* F & & \\ \downarrow & & \downarrow \mu \\ f_1^* d_1^* E \cong d_1^* f_0^* E \cong d_1^* F & & \end{array}$$

Since  $f_1^*$  is full and faithful by assumption, this arrow comes from a unique arrow  $\sigma : d_0^*E \rightarrow d_1^*E$ . This arrow  $\sigma$  satisfies the cocycle condition in  $\mathcal{E}_2$  because in  $\mathcal{F}_2$ , the map  $\mu$ , and hence also  $f_1^*(\sigma)$ , does, while  $f_2^* : \mathcal{E}_2 \rightarrow \mathcal{F}_2$  is faithful by assumption. The arrow  $\sigma$  also satisfies the unit condition in  $\mathcal{E}_0$  for a similar reason, since  $f_0^* : \mathcal{E}_0 \rightarrow \mathcal{F}_0$  is again faithful (in fact  $f_0 : \mathcal{E}_0 \rightarrow \mathcal{F}_0$  is a retract of  $f_1 : \mathcal{E}_1 \rightarrow \mathcal{F}_1$ , so  $f_0$  is connected since  $f_1$  is). This shows that from an object (3.4) in the pushout  $\mathcal{P}$ , one can construct an object in  $\mathcal{D}(\mathcal{E})$ .

Conversely, any object  $(E, \sigma)$  in  $\mathcal{D}(\mathcal{E})$  gives an object  $(F, \mu, E, v)$  in the pushout, where  $F = f_0^*E$  and  $\mu$  is defined as

$$\mu : d_0^*F = d_0^*f_0^*E \cong f_1^*d_0^*E \xrightarrow{f_1^*\sigma} f_1^*d_1^*E \cong d_1^*f_0^*E \cong d_1^*F,$$

and  $v : F \rightarrow f_0^*E$  is defined to be the identity.

These constructions establish a suitable equivalence of categories  $\mathcal{D}(\mathcal{E}) \cong \mathcal{P}$ , proving the lemma.  $\square$

#### 4. The topos defined by an atlas

This section is of auxiliary character. It defines a topos  $\text{sh}(M, \mathcal{U})$  out of an atlas  $\mathcal{U}$  for a local equivalence relation  $r$  on the locale  $M$ , and  $\text{sh}(M, \mathcal{U})$  in general will depend on the choice of  $\mathcal{U}$  (and even, in the most general case, on some further choice of a ‘hypercovering’).

For any atlas  $\mathcal{U} = \{(U_j, R_j)\}_{j \in I}$  for a local equivalence relation, we construct a simplicial locale  $U$ . (a hypercovering of  $M$ , in fact): the locale  $U_0$  of vertices is the disjoint sum

$$U_0 = \coprod_{j \in I} U_j, \quad (4.1)$$

while the space  $U_1$  of 1-simplices is defined as

$$U_1 = \coprod_{i, j \in I} \coprod_{k \in K_{i, j}} U_{ijk}, \quad (4.2)$$

where  $K_{i, j}$  is an index set for some open covering  $U_{ijk}$  of  $U_i \cap U_j$  by sublocales on which  $R_i$  and  $R_j$  agree. The simplicial operators

$$U_1 \rightrightarrows U_0 \quad (4.3)$$

are defined in the obvious way (if we assume, as we may, that  $K_{ii} = \{*\}$ , a one-point set, and that  $U_{j, j, *} = U_j$ ). We now define  $U$ . as the coskeleton of the truncated simplicial locale (4.3).

$$U = \text{Cosk}(U_1 \rightrightarrows U_0). \quad (4.4)$$

Thus,  $U_2$  is a coproduct with an index set whose typical element is given by data  $((i_0, i_1, i_2), (k_0, k_1, k_2))$  with the  $i$ 's in  $I$ , and  $k_2 \in K_{i_0 i_1}$ , etc., and the summand corresponding to this index is

$$U_{i_0 i_1 k_2} \cap U_{i_1 i_2 k_0} \cap U_{i_0 i_2 k_1}.$$

The simplicial locale  $U$ . has an evident augmentation  $a$  to  $M$  given by the inclusions  $U_i \rightarrow M$  ( $i \in I$ ). All maps in the diagram

$$\cdots U_2 \rightrightarrows U_1 \rightrightarrows U_0 \rightarrow M \quad (4.5)$$

are étale, so  $U$ . is a simplicial sheaf on  $M$ .

**Lemma 4.1.** *The descent topos  $\mathcal{D}(\text{sh}(U))$  is equivalent to the topos  $\text{sh}(M)$  of sheaves on  $M$ , by an equivalence compatible with the augmentations (3.1) and (4.5).*

**Proof.** We view  $U$ . as a simplicial sheaf on  $M$ . Since  $U_1 \rightarrow U_0 \times U_0$  is surjective and  $U$ . is defined as a coskeleton,  $U$ . is clearly a hypercover of  $M$  (i.e. an internal contractible simplicial set inside  $\text{sh}(M)$ ). By standard theory of simplicial covering spaces [5, Appendix] applied in  $\text{sh}(M)$ , an object of  $\mathcal{D}(\text{sh}(U))$  can be identified with a covering projection into  $U$ .. But by contractibility of  $U$ ., each such is a trivial covering projection, i.e. it corresponds to a sheaf of  $M$ . This proves the lemma.  $\square$

The sum of the equivalence relations  $R_j$  defines an equivalence relation  $R_0$  on the sum  $U_0$  (cf. (4.1)); similarly, the sum of the equivalence relations

$$R_i|_{U_{ijk}} = R_j|_{U_{ijk}}$$

defines an equivalence relation  $R_1$  on the sum  $U_1$  (cf. (4.2)), and on  $U_2$ , etc. By the evident compatibilities, we get a morphism of simplicial locales

$$q_n : U_n \rightarrow U_n / R_n \quad (n = 0, 1, 2, \dots)$$

and hence a morphism of their respective descent topoi; we denote the descent topos for the simplicial topos  $(\text{sh}(U_n / R_n))_n$  by  $\text{sh}(M, U)$ . All this is depicted in the following diagram (utilizing Lemma 4.1 for the descent of the left-hand column):

$$\begin{array}{ccc} \begin{array}{c} \downarrow \downarrow \downarrow \\ \text{sh}(U_1) \end{array} & \longrightarrow & \begin{array}{c} \downarrow \downarrow \downarrow \\ \text{sh}(U_1 / R_1) \end{array} \\ \downarrow & & \downarrow \\ \begin{array}{c} \downarrow \downarrow \\ \text{sh}(U_0) \end{array} & \longrightarrow & \begin{array}{c} \downarrow \downarrow \\ \text{sh}(U_0 / R_0) \end{array} \\ \downarrow & & \downarrow \\ \text{sh}(M) & \longrightarrow & \text{sh}(M, U) \end{array} \quad (4.6)$$

**Lemma 4.2.** *For any open atlas  $\mathcal{U}$  and choice of hypercovering  $U$ , the topos  $\mathrm{sh}(M, U)$  is an étendue.*

**Proof.** If  $(W, R)$  is an open chart, and  $V \subseteq W$  is an open sublocale, then one obtains an inclusion of an open sublocale  $V/(R|_V) \rightarrow W/R$ . In the right-hand column of (4.6) each map  $U_n/R_n \rightarrow U_{n-1}/R_{n-1}$  is a sum of such inclusions, hence is étale. By Lemma 3.2, the descent topos  $\mathrm{sh}(M, U)$  is an étendue.  $\square$

The situation simplifies considerably for the case where  $r$  is an (open and) locally connected local equivalence relation. If  $\mathcal{U}$  is a connected atlas for  $r$ , we may choose the  $U_{ijk}$  so small that the charts  $(U_{ijk}, R_i)$  are all connected (and open, locally connected, of course). If this is the case, we say that the hypercovering  $U$  is connected; then the geometric morphisms

$$\mathrm{sh}(U_n) \rightarrow \mathrm{sh}(U_n/R_n) \quad (n = 0, 1)$$

are connected geometric morphisms. Consequently, we have by Lemmas 3.5 and 3.4, the following lemma:

**Lemma 4.3.** *For a connected atlas  $\mathcal{U}$  and any connected hypercovering  $U$  associated to it,*

$$\begin{array}{ccc} \mathrm{sh}(U_0) & \longrightarrow & \mathrm{sh}(U_0/R_0) \\ \downarrow & & \downarrow \\ \mathrm{sh}(M) & \xrightarrow{\pi} & \mathrm{sh}(M, U) \end{array} \quad (4.7)$$

*is a push-out, and the geometric morphism  $\pi : \mathrm{sh}(M) \rightarrow \mathrm{sh}(M, U)$  is again connected.*  $\square$

(The geometric morphism  $\pi$  is also locally connected.)

Thus,  $\mathrm{sh}(M, U)$  may be identified, via  $\pi^*$ , with a full subcategory of  $\mathrm{sh}(M)$ , and since the remaining parts of the diagram (4.7) do not depend on the choice of  $U$ , it follows that  $\mathrm{sh}(M, U)$  only depends on the atlas  $\mathcal{U}$  itself, not on the choice of hypercovering  $U$ , as long as  $U$  is taken to be connected. Therefore, we may write  $\mathrm{sh}(M, \mathcal{U})$  for  $\mathrm{sh}(M, U)$ . It is an étendue, by Lemma 4.2. The objects of  $\mathrm{sh}(M, \mathcal{U})$  we call  $\mathcal{U}$ -sheaves.

We already observed that for  $r$  locally connected,  $\mathrm{sh}(M, r)$  is a full subcategory of  $\mathrm{sh}(M)$ , so we may compare it with the  $\mathrm{sh}(M, \mathcal{U})$ 's. It is clear from Proposition 1.1 that if the structure of  $r$ -sheaf on a sheaf  $E$  is given by an atlas  $\mathcal{U}$ , then  $E \in \mathrm{sh}(M, \mathcal{U})$ ; and conversely, every  $\mathcal{U}$ -sheaf is an  $r$ -sheaf, so that  $\mathrm{sh}(M, r)$  is the union of all the subcategories  $\mathrm{sh}(M, \mathcal{U})$  as  $\mathcal{U}$  ranges over the connected atlases for  $r$ . This union is actually a filtered one; for, any two connected atlases for  $r$

have a common refinement, and it is easy to see that if  $\mathcal{U}'$  refines  $\mathcal{U}$ , then  $\text{sh}(M, \mathcal{U}) \subseteq \text{sh}(M, \mathcal{U}')$ .

## 5. Simply connected maps and étendues

Let  $f : Y \rightarrow X$  be a map of locales. We shall call  $f$  *simply connected* if

- (i)  $f$  is connected (i.e.  $f^* : \text{sh}(X) \rightarrow \text{sh}(Y)$  is full and faithful),
- (ii) for every sheaf  $E$  on  $Y$ , if there exists an open cover  $\bigcup U_i = Y$  of  $Y$  and sheaves  $D_i$  on  $X$  such that  $E|_{U_i} \cong f^*(D_i)|_{U_i}$ , then there exists a sheaf  $D$  on  $X$  such that  $E \cong f^*(D)$ .

Condition (ii) expresses that if a sheaf  $E$  on  $Y$  is locally in the image of  $f^*$ , then it is in the image of  $f^*$  (up to isomorphism). (The conditions together express the intuitive idea that  $f$  is a map with simply connected fibers, in a very weak way, but sufficient for our purposes in this paper. Surely for a general theory of simply connected maps, one should use a stronger notion, which is stable under pullback.)

**Examples.** (a) The unique map  $Y \rightarrow 1$  is simply connected iff every locally constant sheaf on the locale  $Y$  is constant. In particular, if a path-connected topological space  $T$  is simply connected in the usual sense (defined in terms of paths), then the unique map  $T \rightarrow 1$  is simply connected.

(b) If  $T \rightarrow B$  is a locally connected map of topological spaces with connected and simply connected fibers (in the usual topological sense), then as a map of locales,  $f$  is simply connected in the sense just defined. (This is not trivial; a detailed proof is given in [10, Lemma 3.2] and [11].)

(c) The standard argument that a locally constant sheaf on the (localic) unit interval  $I$  is constant will (when applied internally in  $\text{sh}(X)$ ) show that the projection  $X \times I \rightarrow X$  is simply connected, for every locale  $X$ .

(d) Let  $Y$  be a connected and locally connected locale, and suppose that the restriction map  $Y^\Delta \rightarrow Y^{\partial\Delta}$  is a stable surjection (here  $\Delta$  is the standard 2-simplex, and  $\partial\Delta$  is its boundary). Then example (c) and [6, Lemma 3.4] show that the map  $Y \rightarrow 1$  is simply connected.

(e) The previous example can be relativized: A connected and locally connected map of locales  $f : Y \rightarrow X$  is simply connected, in the sense defined above, whenever  $(Y^\Delta)_X \rightarrow (Y^{\partial\Delta})_X$  is a stable surjection. (Here, for any locale  $A$ ,  $(Y^A)_X$  denotes the relative exponential 'of maps  $A \rightarrow Y$  which become constant when composed with  $f : Y \rightarrow X$ ', i.e. the locale defined by the pull-back diagram

$$\begin{array}{ccc} (Y^A)_X & \longrightarrow & Y^A \\ \downarrow & & \downarrow f^A \\ X & \longrightarrow & X^A \end{array}$$

where the map  $X \rightarrow X^A$  is the exponential adjoint of the projection map  $X \times A \rightarrow A$ .)

The notion of simply connected map given here is related to local equivalence relations in the following way. For any open map  $f: Y \rightarrow X$ , its kernel pair  $\text{Ker}(f) = Y \times_X Y \subseteq Y \times Y$  defines an equivalence relation on  $Y$ . The induced local equivalence relation on  $Y$ , given by the atlas consisting of the single chart  $(Y, \text{Ker}(f))$ , is called the *local kernel pair* of  $f$ , and denoted  $\text{Lker}(f)$ . If the map  $f$  is locally connected, then so is this local equivalence relation on  $Y$ , and we have a category  $\text{sh}(Y; \text{Lker}(f))$ , together with an evident factorization of  $f^*: \text{sh}(X) \rightarrow \text{sh}(Y)$  through the forgetful functor  $\text{sh}(Y; \text{Lker}(f)) \rightarrow \text{sh}(Y)$ .

The following is now obvious from the definition, and from Proposition 1.1.

**Lemma 5.1.** *A locally connected map  $f: Y \rightarrow X$  is simply connected iff  $f^*$  induces an equivalence of categories  $\text{sh}(X) \approx \text{sh}(Y; \text{Lker}(f))$ .  $\square$*

An equivalence relation  $R$  on a locale  $X$  is said to be *simply connected* if the quotient map  $X \rightarrow X/R$  is a simply connected map. (If  $d_0, d_1: R \rightarrow X$  are locally connected, it can be shown that  $X \rightarrow X/R$  is simply connected whenever  $d_0, d_1$  are; but we will neither use nor prove this here.) Moreover, an atlas for a local equivalence relation is called simply connected if all its charts are; and a local equivalence relation  $r$  is called *locally simply connected* if every atlas for  $r$  can be refined by a simply connected atlas (this implies that  $r$  is locally connected).

**Lemma 5.2.** *Let  $r$  be a locally connected local equivalence relation on the locale  $M$ , and let  $\mathcal{U}$  be a simply connected atlas for  $r$ . Then the inclusion functor  $\text{sh}(M, \mathcal{U}) \rightarrow \text{sh}(M, r)$  is an equivalence of categories.*

**Proof.** The inclusion functor is a functor between full subcategories of  $\text{sh}(M)$ , hence is full and faithful. To see that it is essentially surjective, consider a sheaf  $E$  on  $M$  with  $r$ -transport. We have to show that there exists an atlas for this  $r$ -transport with underlying  $r$ -atlas the given atlas  $\mathcal{U}$ . By the uniqueness of transport, this means that we have to show that for any chart  $(U, R)$  of  $\mathcal{U}$ , the restricted sheaf  $E|_U$  is isomorphic to  $\pi^*(D)$  for some sheaf  $D$  on  $U/R$  (where  $\pi$  is the quotient map  $U \rightarrow U/R$ ). Since  $E$  has  $r$ -transport, there exists an atlas  $\mathcal{V}$  for  $r$ , whose charts  $(V_j, R_j)$  act on  $E|_{V_j}$ , so for the given  $U$ , there exists a covering  $\bigcup V_i$  of  $U$  such that for each index  $i$  there exist a sheaf  $D_i$  on  $V_i/R_i$  with  $E|_{V_i} \cong \pi_i^*(D_i)$ , where  $\pi_i: V_i \rightarrow V_i/R_i$  is the quotient map. Let  $\mu_i: V_i \rightarrow U$  and  $\nu_i: V_i/R_i \rightarrow U/R$  be the inclusions, so that  $\nu_i \pi_i = \pi \mu_i$ . Then  $D_i \cong \nu_i^* \nu_{i*}(D_i)$ , so  $E|_{V_i} \cong \pi_i^* \nu_i^* (\nu_{i*} D_i) \cong \mu_i^* \pi^* (\nu_{i*} D_i) \cong \pi^* (\nu_{i*} D_i)|_{V_i}$ . Thus  $E|_{V_i}$  is in the image of  $\pi^*$ , up to isomorphism. Since by assumption the quotient map  $\pi: U \rightarrow U/R$  is simply connected, it follows that  $E|_U$  is isomorphic to  $\pi^*(D)$  for some sheaf  $D$  on  $U/R$ , as required.  $\square$

This lemma, together with Lemma 4.2, yields the following theorem:

**Theorem 5.3.** *Let  $r$  be a local equivalence relation on a locale  $M$ . If  $r$  is locally simply connected, then  $\text{sh}(M, r)$  is an étendue topos.  $\square$*

## 6. Maps from locales into étendues

Let  $\mathcal{T}$  be a fixed étendue topos. In this section, we will show how for any locale  $M$ , a locally connected geometric morphism  $a : \text{sh}(M) \rightarrow \mathcal{T}$  gives rise to a local equivalence relation on  $M$ . Recall that for locally connected  $a$ , the inverse image functor  $a^*$  has a left adjoint  $a_! : \text{sh}(M) \rightarrow \mathcal{T}$ .

**Lemma 6.1.** *The locale  $M$  has a basis of open sublocales  $U \subseteq M$  with the property that  $\mathcal{T}/a_!U$  is a localic topos.*

**Proof.** Let  $G$  be an object of  $\mathcal{T}$  for which  $\mathcal{T}/G$  is a localic topos, and construct the locale  $B$  by the pull-back

$$\begin{array}{ccc} \text{sh}(B) & \longrightarrow & \mathcal{T}/G \\ q \downarrow & & \downarrow p \\ \text{sh}(M) & \xrightarrow{a} & \mathcal{T} \end{array}$$

Then, by construction of  $B$ , the topos  $\text{sh}(B)$  is equivalent over  $\text{sh}(M)$  to  $\text{sh}(M)/a^*(G)$ . And  $q$  is induced by an étale map (a local homeomorphism) of locales, also denoted  $q : B \rightarrow M$ . The required basis for  $M$  consists of those open  $U \subseteq M$  over which  $q$  has a section. Indeed, let  $s : U \rightarrow B$  be a section of  $q$ . This section can be viewed as a map  $s : U \rightarrow a^*(G)$  in  $\text{sh}(M)$ , and hence corresponds by adjunction to a map  $\hat{s} : a_!(U) \rightarrow G$  in  $\mathcal{T}$ . But then the topos  $\mathcal{T}/a_!(U) = (\mathcal{T}/G)/\hat{s}$  is localic since  $\mathcal{T}/G$  is.  $\square$

By the lemma, any open  $U \subseteq M$  in this basis for  $M$  gives rise to a locale  $a_\#(U)$  and a map  $\varepsilon_U : U \rightarrow a_\#(U)$ , for which there is an equivalence of topoi under  $\text{sh}(U)$ , as in

$$\begin{array}{ccc} \text{sh}(U) & \xrightarrow{\varepsilon_U} & \text{sh}(a_\# U) \approx \mathcal{T}/a_!(U) \\ \downarrow & & \downarrow \\ \text{sh}(M) & \longrightarrow & \mathcal{T} \end{array}$$

Notice that by construction,  $\varepsilon_U$  is a connected and locally connected map of locales. Thus, since connected locally connected maps are stable under pullback,

$$R_U := \text{Ker}(\varepsilon_U) \subseteq U \times U$$

is a connected and locally connected equivalence relation on  $U$ . We shall prove the following:

**Lemma 6.2.** *The charts  $(U, R_U)$ , for all open  $U \subseteq M$  for which  $\mathcal{T}/a_!(U)$  is localic, form an atlas for a local equivalence relation on  $M$ .*

We will call this local equivalence relation the *local kernel* of  $a$ , and denote it by  $\text{Lker}(a)$ . (This is compatible with the similar notation used in Section 5.) Clearly  $\text{Lker}(a)$  is locally connected.

**Proof.** For two such open  $V \subseteq U \subseteq M$ , it is enough to show that  $\text{Lker}(\varepsilon_U)|_V = \text{Lker}(\varepsilon_V)$ . Consider the diagram

$$\begin{array}{ccc} V & \xrightarrow{i} & U \\ \varepsilon_V \downarrow & & \downarrow \varepsilon_U \\ a_{\#}V & \xrightarrow{a_{\#}(i)} & a_{\#}U \end{array}$$

obtained from the inclusion  $i : V \subseteq U$ . Since  $\mathcal{T}/a_!V \rightarrow \mathcal{T}/a_!U$  is a map of slice topoi over  $\mathcal{T}$ , the corresponding map of locales  $a_{\#}V \rightarrow a_{\#}U$  is étale. Thus

$$\begin{aligned} \text{Lker}(\varepsilon_U)|_V &= \text{Lker}(\varepsilon_U i) && \text{(since } i \text{ is an open inclusion)} \\ &= \text{Lker}(a_{\#}(i) \varepsilon_V) \\ &= \text{Lker}(\varepsilon_V), \end{aligned}$$

where the latter equality holds by the following lemma.

**Lemma 6.3.** *Let  $f : Y \rightarrow X$  and  $e : X \rightarrow B$  be maps of locales, where  $e$  is étale. Then  $\text{Lker}(f) = \text{Lker}(ef)$ .*

**Proof.** Consider an open  $U \subseteq X$  such that  $e|_U$  is a homeomorphism  $U \cong e(U)$ . Then

$$\begin{aligned} \text{Ker}(f)|_{f^{-1}U} &= \text{Ker}(f|_{f^{-1}U}) \\ &= \text{Ker}(ef|_{f^{-1}U}) && \text{(since } e|_U \text{ is an embedding)} \\ &= \text{Ker}(ef)|_{f^{-1}U}. \end{aligned}$$

Since this holds for all such  $U$ ,  $\text{Ker}(f)$  and  $\text{Ker}(ef)$  agree on an open cover of  $Y$ . Hence  $\text{Lker}(f) = \text{Lker}(ef)$ .  $\square$

This construction of the local equivalence relation  $\text{Lker}(a)$  on  $M$  from the



locally connected geometric morphism  $a : \text{sh}(M) \rightarrow \mathcal{T}$  enjoys various naturality properties. We single out the following. Recall the  $\#$ -construction of Section 2 for lifting a local equivalence back along an étale map. Then the following holds:

**Lemma 6.4.** *For a pull-back square*

$$\begin{array}{ccc} \text{sh}(N) & \xrightarrow{b} & \mathcal{R} \\ q \downarrow & & \downarrow p \\ \text{sh}(M) & \xrightarrow{a} & \mathcal{T} \end{array}$$

where  $a$  (and hence  $b$ ) are locally connected geometric morphisms and  $p$  is an étale map between étendues, we have  $q^\#(\text{Lker}(a)) = \text{Lker}(b)$ .

**Proof.** Let  $V \subseteq N$  be an open sublocale of  $N$ , so small that  $q|_V$  is a homeomorphism  $V \cong q(V)$ , and moreover so small that both  $\mathcal{R}/b|_V$  and  $\mathcal{T}/a|(q(V))$  are localic topoi. (Note that this property is inherited by smaller open sublocales.) Then there is an induced map  $b_\#(V) \rightarrow a_\#(q(V))$  which is étale (in fact a homeomorphism) since  $p$  is. The result follows from Lemma 6.3.  $\square$

**Remark 6.5** (which we shall not use). If  $r$  is a local equivalence relation on a locale  $M$ , and  $\mathcal{U}$  is a connected atlas for  $r$ , then there is an induced geometric morphism  $a : \text{sh}(M) \rightarrow \text{sh}(M, \mathcal{U})$ , as in Section 4. The local equivalence relation  $\text{Lker}(a)$  is in general larger than  $r$ . It coincides with  $r$  if  $r$  has an atlas consisting of charts  $(U, R)$  with the property that  $R$  is the kernel pair of  $U \rightarrow U/R$ .

## 7. The main theorem

We now prove the result announced in the title of the paper:

**Theorem 7.1.** *For every étendue  $\mathcal{T}$ , there exists a locale  $M$  and a locally simply connected local equivalence relation  $r$  on  $M$  for which there exists an equivalence of topoi  $\text{sh}(M, r) \approx \mathcal{T}$ .*

In the proof, we shall use the following construction from [6]: for any topos  $\mathcal{E}$ , there exists a locale  $X = X_{\mathcal{E}}$  in  $\mathcal{E}$  such that  $X$  is (internally) contractible and locally contractible, and moreover such that the topos  $\mathcal{E}[X]$  of  $\mathcal{E}$ -internal sheaves on  $X$  is (externally) localic. In particular, this locale  $X$  has (internally in  $\mathcal{E}$ ) a basis, containing  $X$  itself, and consisting of open  $U \subseteq X$  which are connected and locally connected, and ‘simply connected’ in the sense that  $U^\Delta \rightarrow U^{\delta\Delta}$  is a stable surjection of locales in  $\mathcal{E}$  (cf. Example (d) in Section 5 for notation). Moreover, these properties of the internal locale  $X$  are stable under pull-back along an

arbitrary geometric morphism  $f : \mathcal{F} \rightarrow \mathcal{E}$ . In the special case where  $f : \mathcal{F} \rightarrow \mathcal{E}$  is such that both  $\mathcal{F}$  and  $\mathcal{F}[f^{\#}(X_{\mathcal{E}})]$  are localic, then  $\mathcal{F}[f^{\#}(X_{\mathcal{E}})] \rightarrow \mathcal{F}$  corresponds to an (external) map of locales  $b : B \rightarrow A$ , and the stable internal properties of  $X$  just listed can be rephrased as follows:  $b$  is an open surjection, and  $B$  has a basis, containing  $B$  itself, which consists of open sublocales  $U \subseteq B$  with the property that each restriction  $b|_U : U \rightarrow b(U)$  satisfies the conditions for the map  $f : T \rightarrow X$  in Example (e) in Section 5; in particular, each such restriction is a simply connected map.

For the proof of the theorem, consider for the given étendue  $\mathcal{T}$  such a locale  $X = X_{\mathcal{T}}$  in  $\mathcal{T}$ . Since  $\mathcal{T}[X_{\mathcal{T}}]$  is localic, there is a locale  $M$  and a geometric morphism  $a : \text{sh}(M) \rightarrow \mathcal{T}$  for which  $\text{sh}(M) \approx \mathcal{T}[X_{\mathcal{T}}]$ , over  $\mathcal{T}$ . Moreover,  $a$  is a connected and locally connected geometric morphism, since  $X_{\mathcal{T}}$  is a connected and locally connected locale in  $\mathcal{T}$ .

Now let  $G$  be an object with full support in  $\mathcal{T}$  for which  $\mathcal{T}/G$  is a localic topos (such a  $G$  exists since  $\mathcal{T}$  is an étendue). Thus there is a locale  $A$  and an étale surjection  $p : \text{sh}(A) \rightarrow \mathcal{T}$  such that  $\mathcal{T}/G \approx \text{sh}(A)$ , over  $\mathcal{T}$ . It follows that the pull-back of  $p$  along  $a : \text{sh}(M) \rightarrow \mathcal{T}$  is again étale. Hence this pull-back is a localic topos, say  $\text{sh}(B)$  as in

$$\begin{array}{ccc} \text{sh}(B) & \xrightarrow{b} & \text{sh}(A) \\ q \downarrow & & \downarrow p \\ \text{sh}(M) & \xrightarrow{a} & \mathcal{T} \end{array} \quad (7.1)$$

The local equivalence relation  $r$  on  $M$  in the statement of the theorem will be  $\text{Lker}(a)$ , as constructed in Section 6. A comparison of the pushout squares in the following two lemmas will now prove the equivalence of topoi  $\text{sh}(M, r) \approx \mathcal{T}$  asserted in the theorem.

**Lemma 7.2.** *The pull-back square (7.1) of topoi is also a pushout square.*

**Proof.** The maps  $p$  and  $q$  are étale surjections, hence descent maps [7, 13]. In other words,  $\text{sh}(M)$  is obtained by descent from the simplicial topos of sheaves on the simplicial locale

$$B_{\bullet} = (\cdots B \times_M B \times_M B \rightrightarrows B \times_M B \rightrightarrows B),$$

and  $\mathcal{T}$  is similarly obtained from the simplicial topos

$$\text{sh}(A_{\bullet}) = (\cdots \text{sh}(A) \times_{\mathcal{T}} \text{sh}(A) \rightrightarrows \text{sh}(A) \rightrightarrows \text{sh}(A)).$$

Moreover, all the components of the simplicial map  $\text{sh}(B_{\bullet}) \rightarrow \text{sh}(A_{\bullet})$  are pull-

backs of the map  $a : \text{sh}(M) \rightarrow \mathcal{T}$ , hence are connected and locally connected. The lemma thus follows from Lemma 3.5.  $\square$

**Lemma 7.3.** *With  $A, B$ , etc. as above, there is a pushout square of topoi*

$$\begin{array}{ccc} \text{sh}(B) & \xrightarrow{b} & \text{sh}(A) \\ q \downarrow & & \downarrow \\ \text{sh}(M) & \longrightarrow & \text{sh}(M, \text{Lker}(a)) \end{array}$$

**Proof.** Note first that by the properties of the [6]-construction listed above, the map  $b : B \rightarrow A$  is connected, locally connected, and simply connected. Hence by Lemma 5.1, the map  $b : B \rightarrow A$  induces an equivalence of categories  $\text{sh}(A) \approx \text{sh}(B, \text{Lker}(b))$ . In particular, the latter is a topos. Write  $\mathcal{P}$  for the pushout of  $\text{sh}(B) \rightarrow \text{sh}(B, \text{Lker}(b))$  and  $q : \text{sh}(B) \rightarrow \text{sh}(M)$ . By the explicit description of pushouts of topoi given in Section 3,  $\mathcal{P}$  is the category of triples  $(F, F', \sigma)$ , where  $F'$  is a  $\text{Lker}(b)$ -sheaf on  $B$ ,  $F$  is a sheaf on  $M$ , and  $\sigma$  is an isomorphism  $q^*(F) \cong F'$  of sheaves on  $B$ . In other words,  $\mathcal{P}$  is (equivalent to) the category of sheaves  $F$  on  $M$  such that  $q^*(F)$  is an  $\text{Lker}(b)$ -sheaf. But, by Lemma 6.4,  $\text{Lker}(b) = q^* \text{Lker}(a)$ , so by Remark 2.1, we conclude that the pushout  $\mathcal{P}$  is equivalent to the category  $\text{sh}(M, \text{Lker}(a))$ . This proves the lemma.  $\square$

Note that we have proved that  $\text{sh}(M, \text{Lker}(a))$  is a topos without invoking Theorem 5.3 and the (as yet unproved) fact that  $\text{Lker}(a)$  is locally simply connected, as we asserted in the theorem. To prove this fact, it suffices to show that  $\text{Lker}(b)$  is locally simply connected, since  $q : B \rightarrow M$  is étale and  $\text{Lker}(b) = q^* \text{Lker}(a)$ . But as explained above, the properties of the [6]-construction imply that  $B$  has a basis of open sublocales  $U \subseteq B$  for which  $U \rightarrow b(U)$  is a connected, locally connected, and simply connected map, so that  $\text{Ker}(b|_U)$  is a simply connected (and locally connected) equivalence relation on  $U$ . Since the collection of these  $U \subseteq B$  form a basis for  $B$ , the local equivalence relation  $\text{Lker}(b)$  must be locally simply connected.

This completes the proof of the theorem.  $\square$

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